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On a class of exponential-type operators and their limit semigroups

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Abstract

The paper is mainly focused upon the study of a class of second order degenerate elliptic operators on unbounded intervals.

We show that these operators generate strongly continuous semigroups in suitable weighted spaces of continuous functions.

Furthermore, we represent the semigroups as limits of iterates of the so-called exponential-type operators.

In a particular case, starting from the stochastic differential equations associated with these operators, we also find an integral representation of the semigroup and determine its asymptotic behaviour.

1. Introduction

In the papers [4,6,10] the authors showed, among other things, that the iterates of Szász–Mirakjan operators, Baskakov operators and Post–Widder operators converge to C_0-semigroups of positive operators acting on suitable weighted spaces of continuous functions on $[0, +\infty[$.

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The generators of these semigroups are showed to be the differential operators $A_i(u) = \frac{1}{2}p_i u''$ defined on suitable domains, where $p_1(x) = x$, $p_2(x) = x(1+x)$ and $p_3(x) = x^2$ $(x \geqslant 0)$.

The three above-mentioned approximation processes fall within a more general class of positive operators, referred to as exponential-type operators, which are generated by an analytic function $p \in \mathcal{C}([0,+\infty[)$ which is strictly positive on $]0,+\infty[$ [11,8].

So, we are naturally led to investigate whether, also in this more general situation, the differential operator $Au = \frac{1}{2}pu''$, defined on a suitable domain, generates a C_0-semigroup of positive operators and whether the semigroup can be represented as a limit of iterates of the exponential-type operators corresponding to p.

We prove that, under suitable assumptions on the growth at infinity of p and its derivatives, the above problem has a positive answer.

In addition, by using results of [2,3], we show that the semigroup is the transition semigroup of a continuous Markov process on $[0,+\infty]$.

In the particular case $p(x) = x^2$ $(x \geqslant 0)$, starting from the stochastic differential equation associated with A, we also find an integral representation of the semigroup and we determine its asymptotic behaviour on bounded continuous functions.

2. Preliminaries on exponential-type operators

Throughout the paper we shall denote by $\mathcal{C}([0,+\infty[)$ the space of all real-valued continuous functions on $[0,+\infty[$ and by $\mathcal{C}_b([0,+\infty[)$ the Banach space of all bounded continuous functions on $[0,+\infty[$, endowed with the sup-norm $\|\cdot\|_\infty$.

The symbol $UC_b^2([0,+\infty[)$ will stand for the space of all functions $f \in \mathcal{C}^2([0,+\infty[)$ such that f'' is uniformly continuous and bounded.

We shall also consider the following closed subspaces of $\mathcal{C}_b([0,+\infty[)$:

$$\mathcal{C}_0([0,+\infty[) := \{f \in \mathcal{C}([0,+\infty[) \mid \lim_{x\to+\infty} f(x) = 0\}$$

and

$$\mathcal{C}_*([0,+\infty[) := \{f \in \mathcal{C}([0,+\infty[) \mid \lim_{x\to+\infty} f(x) \in \mathbb{R}\}.$$

For any $m \geqslant 1$ set

$$w_m(x) = \frac{1}{1+x^m}, \quad (x \geqslant 0), \tag{2.1}$$

$$E_m := \{f \in \mathcal{C}([0,+\infty[) \mid w_m f \in \mathcal{C}_b([0,+\infty[)\} \tag{2.2}$$

and

$$E_m^0 := \{f \in \mathcal{C}([0,+\infty[) \mid w_m f \in \mathcal{C}_0([0,+\infty[)\}. \tag{2.3}$$

The space E_m will be endowed with the Banach norm

$$\|f\|_m := \|w_m f\|_\infty \quad (f \in E_m). \tag{2.4}$$

E_m^0 is a closed subspace of E_m. Moreover, for every $m \geqslant 1$,

$$\mathcal{C}_b\,([0,+\infty[) \subset E_m^0 \subset E_m \subset E_{m+1}^0 \tag{2.5}$$

and

$$\|\cdot\|_m \leqslant \|\cdot\|_\infty \text{ on } \mathcal{C}_b\,([0,+\infty[)\,,\ \|\cdot\|_{m+1} \leqslant \frac{m+2}{m+1}\|\cdot\|_m \text{ on } E_m. \tag{2.6}$$

For the sake of brevity we shall also set $E_0 := \mathcal{C}_b\,([0,+\infty[)$ and $E_\infty := \bigcup_{m=0}^{\infty} E_m$.

From now on we shall fix an analytic function $p \in \mathcal{C}\,([0,+\infty[)$ such that

$$p(0) = 0 \quad \text{and} \quad p(x) > 0 \quad \text{for every } x > 0. \tag{2.7}$$

We shall assume that there exists a family $(\mu_{n,x})_{n\geqslant 1, x\geqslant 0}$ of probability Borel measures on $[0,+\infty[$ such that

$$E_\infty \subset \bigcap_{n\geqslant 1, x\geqslant 0} \mathcal{L}^1(\mu_{n,x}) \tag{2.8}$$

and

$$\frac{d}{dx}\int_0^{+\infty} f(u)\,d\mu_{n,x}(u) = \frac{n}{p(x)}\int_0^{+\infty}(u-x)f(u)\,d\mu_{n,x}(u)$$
$$(n\geqslant 1,\ x>0,\ f\in E_\infty). \tag{2.9}$$

By considering (2.9) as a differential equation in the sense of the theory of generalized functions [15] and by using methods developed in [8] in several cases it is possible to describe a family $(\mu_{n,x})_{n\geqslant 1, x\geqslant 0}$ satisfying (2.8) and (2.9).

Here we present some examples (see [8] for more details):

(i) If $p(x) = x \quad (x\geqslant 0)$, then

$$\mu_{n,x} = \sum_{k=0}^{\infty} e^{-nx}\frac{(nx)^k}{k!}\varepsilon_{k/n},$$

where each $\varepsilon_{k/n}$ denotes the unit mass concentrated at k/n.

(ii) If $p(x) = x(1+x) \quad (x\geqslant 0)$, then

$$\mu_{n,x} = \sum_{k=0}^{\infty}\binom{n+k-1}{k}\frac{x^k}{(1+x)^{n+k}}\varepsilon_{k/n}.$$

(iii) If $p(x) = x^2$, then

$$\mu_{n,x} = \begin{cases} \varepsilon_0, & x = 0,\\ \varphi_{n,x}\lambda_1, & x > 0,\end{cases}$$

where $\varphi_{n,x}(u) = \frac{n^n}{x^n(n-1)!}\exp\left(-\frac{nu}{x}\right)u^{n-1}\ (u\geqslant 0)$ and λ_1 is the Lebesgue measure on $[0,+\infty[$.

(iv) If $p(x) = 2x^{3/2}$ $(x \geqslant 0)$, then

$$\mu_{n,x} = \begin{cases} \varepsilon_0, & x = 0, \\ \exp(-n\sqrt{x})\varepsilon_0 + \psi_{n,x}\lambda_1, & x > 0, \end{cases}$$

where $\psi_{n,x}(u) = \exp(-n\sqrt{x})n \exp(-nu/\sqrt{x})u^{-1/2}I_1(2n\sqrt{u})$ $(u > 1)$ and I_1 is a modified Bessel function of the first kind.

Under assumptions (2.8) we can define a sequence of positive linear operators on E_∞ by setting, for every $n \geqslant 1$ and $f \in E_\infty$,

$$L_n(f)(x) := \int_0^{+\infty} f \, d\mu_{n,x}. \tag{2.10}$$

The operators L_n $(n \geqslant 1)$ are also referred to as the exponential-type operators associated with the function p.

In cases (i), (ii) and (iii) above, they reduce to Szász–Mirakjan operators, Baskakov operators, Post–Widder operators, respectively.

In the sequel we shall also assume that $L_n(f) \in \mathcal{C}([0, +\infty[)$. In particular we get

$$L_n(f) \in \mathcal{C}_b([0, +\infty[) \text{ and } \|L_n(f)\|_\infty \leqslant \|f\|_\infty \text{ for } f \in \mathcal{C}_b([0, +\infty[). \tag{2.11}$$

Setting

$$\psi_x(t) := t - x \quad (t \geqslant 0), \tag{2.12}$$

from (2.9) it follows that, for any $f \in E_\infty$, $L_n(f)$ is differentiable in $]0, +\infty[$ and

$$p(x)L_n(f)'(x) = nL_n(\psi_x f)(x) \quad (x > 0). \tag{2.13}$$

Set $e_k(x) := x^k \quad (x > 0, k \in \mathbb{Z})$. Then from (2.13) it follows that, for every $f \in E_\infty$,

$$L_n(e_1 f)(x) = xL_n(f)(x) + \frac{p(x)}{n}L_n(f)'(x) \quad (x > 0). \tag{2.14}$$

In particular, for $f = \mathbf{1}, e_1, e_2, e_3$, we obtain

$$L_n(\mathbf{1}) = \mathbf{1},\ L_n(e_1) = e_1,\ L_n(e_2) = e_2 + \frac{p}{n}, \tag{2.15}$$

$$L_n(e_3) = e_3 + \frac{3e_1 p}{n} + \frac{pp'}{n^2},$$

$$L_n(e_4) = e_4 + \frac{6e_2 p}{n} + \frac{p(3p + 4e_1 p')}{n^2} + \frac{p((p')^2 + pp'')}{n^3}.$$

As regards the behaviour of the operators on the subspaces E_m we have the following result.

Theorem 2.1. *Assume that*

$$D^r p = O(e_{2-r})\ (x \longrightarrow +\infty) \quad \textit{for every } r \geqslant 0. \tag{2.16}$$

Then

(i) $L_n(E_m) \subset E_m$ *and* $L_n(E_m^0) \subset E_m^0$ *for every* $n \geqslant 1$.
(ii) *Each* L_n *is continuous from* E_m *into itself and* $\|L_n\| \leqslant 1 + \frac{K_m}{n}$, *where* $K_m \geqslant 0$ *is independent on n. Moreover* $K_1 = 0$ *and* $K_2 = \|p\|_2$.

Proof. Let $n \geqslant 1$ be fixed. By using induction on $m \geqslant 1$ we shall prove that

$$L_n(e_m) = e_m + \frac{1}{n}\varphi_m, \tag{2.17}$$

where $\varphi_m \in \mathcal{C}^\infty([0, +\infty[)$ and $D^r\varphi_m = O(e_{m-r})(x \longrightarrow \infty)$ for all $r \geqslant 0$.

According to (2.15), our assertion (2.17) holds true for $m = 1$ and 2.

Suppose that it is true for a given $m \geqslant 2$. By using (2.14) for $f = e_m$ we obtain

$$L_n(e_{m+1}) = e_1 L_n(e_m) + \frac{p}{n} L_n(e_m)'.$$

Thus

$$L_n(e_{m+1}) = e_{m+1} + \frac{1}{n}\varphi_{m+1},$$

where

$$\varphi_{m+1} = e_1\varphi_m + mpe_{m-1} + \frac{1}{n}p\varphi_m'.$$

By using Leibniz's differentiation formula it is easy to verify that for all $r \geqslant 0$,

$$D^r\varphi_{m+1} = O(e_{m+1-r}) \quad (x \longrightarrow \infty).$$

This completes the proof of (2.17).

Let $f \in E_m$. Then $|f| \leqslant \|f\|_m(1 + e_m)$, and so

$$|L_n(f)| \leqslant \|f\|_m(1 + L_n(e_m)) \leqslant \|f\|_m \left(1 + e_m + \frac{1}{n}|\varphi_m|\right).$$

This yields

$$\frac{|L_n(f)|}{1 + e_m} \leqslant \|f\|_m \left(1 + \frac{1}{n}\frac{|\varphi_m|}{1 + e_m}\right),$$

which means that $L_n(f) \in E_m$ and

$$\|L_n(f)\|_m \leqslant \|f\|_m \left(1 + \frac{1}{n}\|\varphi_m\|_m\right).$$

Thus L_n is continuous from E_m into itself and $\|L_n\| \leqslant 1 + \frac{1}{n}K_m$ with $K_m = \|\varphi_m\|_m$.

Now let $f \in E_m^0$ and $\varepsilon > 0$. Then there exists $a \geqslant 0$ such that

$$|f(t)| \leqslant \varepsilon(1 + t^m), \quad t \geqslant a.$$

Set $M = sup\{|f(t)| \mid 0 \leqslant t \leqslant a\}$; there exists $b \geqslant a$ such that

$$M \leqslant \varepsilon(1 + x^m), \quad x \geqslant b.$$

For $x \geqslant b$ we have

$$\begin{aligned}|L_n(f)(x)| &\leqslant \int_0^a |f(t)|\, d\mu_{n,x}(t) + \int_a^\infty |f(t)|\, d\mu_{n,x}(t) \\ &\leqslant M + \varepsilon(1 + L_n(e_m)(x)),\end{aligned}$$

which implies

$$\frac{|L_n(f)(x)|}{1 + x^m} \leqslant \varepsilon \left(3 + \frac{K_m}{n}\right).$$

Thus $L_n(f) \in E_m^0$. $\square$

3. The generator $(A, D_m(A))$

Under the same assumptions of the previous section, for every $m \geqslant 1$ consider the differential operator

$$Au(x) := \begin{cases} \frac{p(x)}{2} u''(x), & x > 0, \\ 0, & x = 0, \end{cases} \tag{3.1}$$

defined on

$$\begin{aligned} D_m(A) := \{u \in E_m^0 \cap \mathcal{C}^2\,(]0, +\infty[) \mid & \lim_{x \to 0^+} p(x)u''(x) \\ &= \lim_{x \to +\infty} w_m(x)p(x)u''(x) = 0\}. \end{aligned} \tag{3.2}$$

Clearly $A(D_m(A)) \subset E_m^0$. Furthermore, we set

$$\begin{aligned} D(\widetilde{A}) := \{u \in \mathcal{C}_*\,([0, +\infty[) \cap \mathcal{C}^2\,(]0, +\infty[) \mid & \lim_{x \to 0^+} p(x)u''(x) \\ &= \lim_{x \to +\infty} p(x)u''(x) = 0\} \end{aligned} \tag{3.3}$$

and

$$\widetilde{A}(u) := A(u) \quad (u \in D(\widetilde{A})). \tag{3.4}$$

Obviously $\widetilde{A}(D(\widetilde{A})) \subset \mathcal{C}_0\,([0, +\infty[)$.

We proceed to show that the operators $(A, D_m(A))$ and $(\widetilde{A}, D(\widetilde{A}))$ are generators of C_0-semigroups of positive operators.

Theorem 3.1. *For every $m \geqslant 1$ the operator $(A, D_m(A))$ is the generator of a C_0-semigroup $(T_m(t))_{t \geqslant 0}$ of positive operators on E_m^0 satisfying $\|T_m(t)\| \leqslant e^{\omega(m,p)t}$ for every $t \geqslant 0$ where $\omega(m, p) := \frac{m(m-1)}{2} \sup_{0 \leqslant x} \frac{x^{m-2} p(x)}{1+x^m}$.*

Moreover, the restrictions of $(T_m(t))_{t\geqslant 0}$ *to the spaces* $\mathcal{C}_0([0,+\infty[)$ *and* $\mathcal{C}_*([0,+\infty[)$ *are Feller semigroups whose generators are* $(\widetilde{A}, D(\widetilde{A}) \cap \mathcal{C}_0([0,+\infty[))$ *and* $(\widetilde{A}, D(\widetilde{A}))$, *respectively.*

Finally, there exists a Markov process $(\Omega, U, (P^x)_{0\leqslant x\leqslant +\infty}, (Z_t)_{0\leqslant t\leqslant +\infty})$ *with state space* $[0,+\infty]$ *and whose paths are continuous almost surely such that for every* $x\geqslant 0$ *and* $t\geqslant 0$

(i) $P^x\{Z_t = +\infty\} = 0$,
(ii) *the distribution* $P^x_{Z_t}$ *of the random variable* Z_t *with respect to* P^x *possesses finite moments of order up to m*,
(iii) $T_m(t)f(x) = \int_\Omega f^*(Z_t)\,dP^x$ *for every* $f \in E^0_m$,

where f^* *denotes the extension of f to* $[0,+\infty]$, *vanishing at* $+\infty$.

Proof. We shall apply Theorems 2.3 and 2.6 of [3] and, to this end, it is enough to verify conditions (2.6), (2.7) and (2.8) of that paper.

In fact, conditions (2.6) and (2.7) are satisfied because of (2.16) (with $r = 0$).
As regards (2.8) we have to show that the following supremum is finite:

$$\omega(m,p) := \sup_{0\leqslant x} \frac{\left|\frac{p(x)}{2}(2w_m'(x)^2 - w_m(x)w_m''(x))\right|}{w_m(x)^2}.$$

A direct calculation yields indeed

$$\omega(m,p) = \frac{m(m-1)}{2} \sup_{0\leqslant x} \frac{x^{m-2}p(x)}{1+x^m}$$

and hence the proof is complete. □

Remark 3.2. As pointed out in [3, p. 219], the Markov process described in Theorem 3.1 depends only on the restriction of the semigroup $(T_m(t))_{t\geqslant 0}$ to $\mathcal{C}_*([0,+\infty[)$ and it is independent of $m\geqslant 1$. Accordingly, the distributions $P^x_{Z_t}$ possess finite moments of any order $m\geqslant 1$ and hence their characteristic functions are infinitely many times continuously differentiable.

Now we proceed to represent the semigroup in terms of iterates of the operators L_n. To this end it is important to find a core for the operator $(A, D_m(A))$.

Recall that if $A : D(A) \subset E \longrightarrow E$ is a linear operator defined on a subspace $D(A)$ of a Banach space E, a subspace D_0 of $D(A)$ is called a *core* for $(A, D(A))$ if D_0 is dense in $D(A)$ with respect to the graph norm

$$\|u\|_A = \|u\| + \|Au\| \quad (u \in D(A)),$$

i.e., for all $u \in D(A)$ and $\varepsilon > 0$, there exists $v \in D_0$ such that $\|u-v\| \leqslant \varepsilon$ and $\|Au-Av\| \leqslant \varepsilon$. If A is closed and if $\lambda I - A$ is invertible for some $\lambda \in \mathbb{C}$, then D_0 is a core for $(A, D(A))$ if and only if $(\lambda I - A)(D_0)$ is dense in E (I stands for the identity operator on E).

We consider the operator $(\widetilde{A}, D(\widetilde{A}))$ described in (3.3) and (3.4). Let

$$D_0 := \{u \in \mathcal{C}_0([0,+\infty[) \cap \mathcal{C}^2([0,+\infty[) \mid \lim_{x\to+\infty} p(x)u''(x) = 0\}. \tag{3.5}$$

Clearly,

$$D_0 \subset D(\widetilde{A}) \cap \mathcal{C}_0([0,+\infty[). \tag{3.6}$$

In the sequel we shall suppose that

$$\lim_{x\to+\infty} p(x) = +\infty, \tag{3.7}$$

$$\text{there exists} \quad a > 0 \text{ such that } ax^2 \leqslant p(x) \quad \text{for every } x \in [0,1], \tag{3.8}$$

$$\text{there exists } \delta_1 > 0 \text{ such that} \quad p \text{ is increasing on } [0,\delta_1]. \tag{3.9}$$

Let us remark that

$$D_0 \subset \{f \in \mathcal{C}_0([0,+\infty[) \cap \mathcal{C}^2([0,+\infty[) \mid \lim_{x\to+\infty} f''(x) = 0\} \subset UC_b^2([0,+\infty[).$$

Indeed, if $u \in D_0$, there exists $M \geqslant 0$ such that $|u''(x)| \leqslant \frac{M}{p(x)}$, $x \geqslant 1$; now from (3.7) one has $\lim_{x\to+\infty} u''(x) = 0$.

Let D_1 be the subspace of $D(\widetilde{A})$ generated by D_0 and the constant function $\mathbf{1}$.

Theorem 3.3. *D_0 is a core for $(\widetilde{A}, D(\widetilde{A}) \cap \mathcal{C}_0([0,+\infty[))$ in $\mathcal{C}_0([0,+\infty[)$, and for $(A, D_m(A))$ in $(E_m^0, \|\cdot\|_m)$, $m \geqslant 1$. Moreover, D_1 is a core for $(\widetilde{A}, D(\widetilde{A}))$ in $\mathcal{C}_*([0,+\infty[)$.*

Proof. Let $u \in D(\widetilde{A}) \cap \mathcal{C}_0([0,+\infty[)$. Due to (3.8), we have

$$\lim_{x\to0^+} x^2 u''(x) = 0.$$

Let $\varepsilon > 0$. Then there exists $\delta > 0$ such that $|u''(x)| \leqslant \frac{\varepsilon}{x^2}$, $x \in]0,\delta]$. Let $0 < x \leqslant min(\delta, \varepsilon/(|u'(\delta)|+1))$. Then

$$|xu'(x)| \leqslant |xu'(x) - xu'(\delta)| + x|u'(\delta)| \leqslant x\int_x^\delta |u''(t)|\,dt + \varepsilon$$
$$\leqslant \varepsilon x\left(\frac{1}{x} - \frac{1}{\delta}\right) + \varepsilon \leqslant 2\varepsilon.$$

This means that

$$\lim_{x\to0^+} xu'(x) = 0.$$

Now, for $\varepsilon > 0$ there exists $\delta \in]0, \delta_1[$ such that

$$|u(z) - u(y)| \leqslant \varepsilon, \quad x|u'(x)| \leqslant \varepsilon,$$
$$x^2|u''(x)| \leqslant \varepsilon, \quad p(x)|u''(x)| \leqslant \varepsilon$$

for all $x, y, z \in]0, \delta[$.
Let $x_0 \in]0, \delta[$. Consider the function

$$v(x) = \begin{cases} u(x_0) + u'(x_0)(x - x_0) + \frac{u''(x_0)}{2}(x - x_0)^2, & 0 \leqslant x \leqslant x_0, \\ u(x), & x > x_0. \end{cases}$$

Then $v \in \mathcal{C}_0([0, +\infty[) \cap \mathcal{C}^2([0, +\infty[)$ and $v \in D_0$. Moreover, for $x \in [0, x_0]$ we have

$$|u(x) - v(x)| \leqslant |u(x) - u(x_0)| + x_0|u'(x_0)| + \frac{|u''(x_0)|}{2}x_0^2 \leqslant \frac{5}{2}\varepsilon$$

and

$$\begin{aligned} |Au(x) - Av(x)| &\leqslant \frac{1}{2}(p(x)|u''(x)| + p(x)|u''(x_0)|) \\ &\leqslant \frac{1}{2}(p(x)|u''(x)| + p(x_0)|u''(x_0)|) \leqslant \varepsilon. \end{aligned}$$

Thus $\|u - v\|_\infty \leqslant \frac{5}{2}\varepsilon$ and $\|Au - Av\|_\infty \leqslant \varepsilon$, which means that D_0 is a core for $(\widetilde{A}, D(\widetilde{A}) \cap \mathcal{C}_0([0, +\infty[))$.

Now let $\lambda > \omega(m, p)$. By Theorem 3.1, $\lambda I - \widetilde{A}$ is invertible and hence $(\lambda I - \widetilde{A})(D_0)$ is dense in $(\mathcal{C}_0([0, +\infty[), \|\cdot\|_\infty)$. On the other hand from the Stone–Weierstrass theorem for weighted spaces, it follows that $\mathcal{C}_0([0, +\infty[)$ is dense in $(E_m^0, \|\cdot\|_m)$ and, obviously, $\|\cdot\|_m \leqslant \|\cdot\|_\infty$ on $\mathcal{C}_0([0, +\infty[)$.

Therefore $(\lambda I - \widetilde{A})(D_0) = (\lambda I - A)(D_0)$ is dense in $(E_m^0, \|\cdot\|_m)$. Since $\lambda I - A$ is invertible, we deduce that D_0 is a core for $(A, D_m(A))$.

Finally, if $u \in D(\widetilde{A})$, set $u(+\infty) := \lim_{x \to +\infty} u(x) \in \mathbb{R}$. Then $u - u(+\infty) \in D(\widetilde{A}) \cap \mathcal{C}_0([0, +\infty[)$. For $\varepsilon > 0$ there exists $v \in D_0$ such that $\|u - u(+\infty) - v\|_\infty \leqslant \varepsilon$ and $\|Au - Av\|_\infty \leqslant \varepsilon$. Consequently, $w := v + u(\infty) \in D_1$ and $\|u - w\|_\infty \leqslant \varepsilon$, $\|Au - Aw\|_\infty \leqslant \varepsilon$. Thus D_1 is a core for $(\widetilde{A}, D(\widetilde{A}))$. □

Before stating the main result, we need the following

Proposition 3.4. *Consider the subspace D_0 described by* (3.5). *Then, for $m \geqslant 2$,*

(i) $\lim_{n \to \infty} n(L_n(u) - u) = \frac{p}{2}u''$ *in $(E_m, \|\cdot\|_m)$ for every $u \in D_0$.*

(ii) $\lim_{n \to \infty} L_n(f) = f$ *in $(E_m^0, \|\cdot\|_m)$ for every $f \in E_m^0$.*

Proof. We shall prove part (i) by applying Proposition 5.1 of [5] (See also [1], Theorem 1). First note that, by using formula (2.15), for every $x \geqslant 0$ we obtain

$$L_n(\psi_x)(x) = 0, \; L_n(\psi_x^2)(x) = \frac{p(x)}{n},$$

$$L_n(\psi_x^3)(x) = \frac{p(x)p'(x)}{n^2},$$

$$L_n(\psi_x^4)(x) = \frac{p(x)}{n}\left[\frac{p'(x)^2 + p(x)p''(x)}{n^2} + \frac{3p(x)}{n}\right],$$

where ψ_x is defined by (2.12).
From these formulae one can easily check that all the assumptions of Proposition 5.1 of [5] are satisfied and so part (i) follows because for every $u \in D_0$ we have $\lim_{x\to\infty} u''(x) = 0$.

As regards part (ii), from (i) it follows that $\lim_{n\to\infty} L_n(u) = u$ in E_m^0, for all $u \in D_0$. On the other hand, the sequence $(L_n)_{n\geqslant 1}$ is equicontinuous due to Theorem 2.1, (ii) and D_0 is dense in$(E_m^0, \|\cdot\|_m)$ since it is a core for $(A, D_m(A))$ and $D_m(A)$ is dense in $(E_m^0, \|\cdot\|_m)$. This proves (ii). □

Theorem 3.5. *Denote by* $(T_m(t))_{t\geqslant 0}$ *the semigroup generated by* $(A, D_m(A))$ *in* E_m^0 $(m\geqslant 2)$. *Then for all* $f \in E_m^0$ *and* $t\geqslant 0$,

$$T_m(t)f = \lim_{n\to\infty} L_n^{k(n)} f \quad \text{in} \quad E_m^0, \tag{3.10}$$

where $(k(n))_{n\geqslant 1}$ *is an arbitrary sequence of positive integers such that* $k(n)/n \longrightarrow t$ *and* $L_n^{k(n)}$ *stands for the iterate of order* $k(n)$ *of* L_n.

In particular, the limit in (3.10) *is uniform on compact subsets of* $[0, +\infty[$.

Proof. From Theorem 2.1 (ii) it follows that for all $n\geqslant 1$ and $p\geqslant 1$,

$$\|L_n^p\| \leqslant \left(1 + \frac{K_m}{n}\right)^p \leqslant \exp\left(K_m \frac{p}{n}\right).$$

Combining this estimate, Theorems 3.1 and 3.3, and Proposition 3.4, the result follows by using a theorem of Trotter ([14, Theorem 5.3]; see also [13, Chapter 3, Theorem 6.7]). □

Remark 3.6. 1. Theorem 3.3 in the cases
$p(x) = x$, L_n = Szász–Mirakjan operators;
$p(x) = x(1+x)$, L_n = Baskakov operators;
$p(x) = x^2$, L_n = Post–Widder operators
was obtained, respectively in [4,6,10].

2. Since E_m^0 is continuously embedded in E_{m+1}^0, from (3.10) it follows that

$$T_{m+1}(t)|_{E_m^0} = T_m(t),\ t\geqslant 0.$$

3. Consider the Markov process described in Theorem 3.1.

Since $L_n(e_1) = e_1$ and

$$L_n(e_2) = e_2 + \frac{p}{n} \leqslant \left(1 + \frac{\|p\|_2}{n}\right) e_2 + \frac{\|p\|_2}{n} \quad (n\geqslant 1)$$

for every $q \geqslant 1$ we obtain

$$L_n^q(e_1) = e_1 \quad \text{and} \quad L_n^q(e_2) \leqslant \left(1 + \frac{\|p\|_2}{n}\right)^q e_2 + \left(1 + \frac{\|p\|_2}{n}\right)^q - 1.$$

Hence from Theorem 3.3 with $m = 2$ we obtain

$$T_2(t)e_1 = e_1,\ T_2(t)e_2 \leqslant \exp(\|p\|_2 t)e_2 + (\exp(\|p\|_2 t) - 1).$$

Therefore, denoted by $E_x(Z_t)$ and $Var_x(Z_t)$ the expected value and the variance of Z_t with respect to P^x ($x \geqslant 0, t \geqslant 0$), by using (i) and (iii) of Theorem 3.1 with $m = 2$, we obtain

$$\begin{aligned} E_x(Z_t) &= T_2(t)(e_1)(x) = x, \\ Var_x(Z_t) &= E_x(Z_t^2) - E_x(Z_t)^2 = T_2(t)(e_2)(x) - x^2 \\ &= (\exp(\|p\|_2 t) - 1)(x^2 + 1). \end{aligned}$$

According to the terminology introduced by Feller ([7]; see also [3, pp. 220–221]), $+\infty$ is a natural boundary point for the process and so, according to Theorem 3.1, (i), as well, the process cannot reach $+\infty$ in a finite time.

The boundary point 0 can be exit or natural according to the behaviour of the function p as $x \to 0^+$.

More precisely, if $\lim\limits_{x \to 0^+} \frac{p(x)}{x^\alpha} \in \mathbb{R} \setminus \{0\} \cup \{+\infty\}$ for some $1 < \alpha < 2$, then 0 is an exit boundary point. In this case the probability that the process located at $]0, +\infty[$ reaches 0 after a finite lapse of time is strictly positive. Moreover, because of the boundary conditions included in the domain $D(\widetilde{A})$, when the process reaches 0 for the first time, it sticks there for ever.

Finally, if $\lim\limits_{x \to 0^+} \frac{p(x)}{x^2} \in \mathbb{R} \setminus \{0\}$, then 0 is a natural boundary point and so it cannot be reached by the process in a finite time.

4. On the semigroup associated with the Post–Widder operators

We start with some introductory remarks. Let

$$\mathcal{K}^2(\mathbb{R}) := \{g \in \mathcal{C}^2(\mathbb{R}) : \ g \text{ has compact support}\}.$$

Consider the evolution problem

$$\begin{cases} \dfrac{\partial u}{\partial t}(x,t) = \dfrac{x^2}{2} \dfrac{\partial^2 u}{\partial x^2}(x,t), & x \in \mathbb{R}, t > 0, \\ u(x,0) = g(x), & x \in \mathbb{R}, \end{cases} \tag{4.1}$$

where $g \in \mathcal{K}^2(\mathbb{R})$.

It corresponds to the differential operator

$$Av(x) = \frac{x^2}{2} v''(x), \quad x \in \mathbb{R}, \ v \in \mathcal{C}^2(\mathbb{R}).$$

(A more general problem, corresponding to the differential operator $(\beta^2/2)x^2 v''(x) + \alpha x v'(x)$, $(\alpha, \beta \in \mathbb{R})$, is presented in [12, Exercise 8.2]).

The stochastic differential equation associated with A is

$$dX_t = X_t \, dB_t, \tag{4.2}$$

where B_t is a one-dimensional Brownian motion starting at 0 (see [12, Definition 2.2.1).

The solution of (4.2) satisfying $X_0 = x \in \mathbb{R}$ is

$$X_t^x = x \exp\left(B_t - \frac{1}{2}t\right), \ t \geqslant 0.$$

(See [12, Exercise 5.6). Consider the function $u(x,t) := Eg(X_t^x), \ t \geqslant 0, \ x \in \mathbb{R}$. By Theorem 8.1.1 in [12], it satisfies (4.1).

Moreover, for $t > 0$ and $x \in \mathbb{R}$ we have

$$u(x,t) = \frac{1}{\sqrt{2\pi t}} \int_{\mathbb{R}} g(xe^{u-t/2}) e^{-u^2/2t} \, du.$$

Now let $\mathcal{K}^2\,(]0,+\infty[) := \{f \in \mathcal{C}^2\,(]0,+\infty[) \mid f \text{ has compact support}\}$ and consider the evolution problem

$$\begin{cases} \dfrac{\partial u}{\partial t}(x,t) = \dfrac{x^2}{2}\dfrac{\partial^2 u}{\partial x^2}(x,t), & x \geqslant 0, \quad t > 0, \\ u(x,0) = f(x), & x \geqslant 0 \end{cases} \tag{4.3}$$

with $f \in \mathcal{K}^2\,(]0,+\infty[)$. Set

$$g(x) = \begin{cases} f(x), & x \geqslant 0, \\ 0, & x < 0. \end{cases}$$

Then $g \in \mathcal{K}^2(\mathbb{R})$ and so $u(x,t) := Eg(X_t^x)$ $(x \in \mathbb{R}, \ t \geqslant 0)$ is a solution of (4.1). It follows that $u(x,t) = Ef(X_t^x)$ provided that $x \geqslant 0, \ t \geqslant 0$ and hence $u(x,t)$ $(x \geqslant 0, t \geqslant 0)$ is a solution of (4.3).

For $t > 0$ we have

$$u(x,t) = \frac{1}{\sqrt{2\pi t}} \int_{\mathbb{R}} f(xe^{u-t/2}) e^{-u^2/2t} \, du.$$

So we are led to consider the operators

$$V(t)f(x) := \frac{1}{\sqrt{2\pi t}} \int_{\mathbb{R}} f(xe^{u-t/2}) e^{-u^2/2t} \, du$$

defined on $\mathcal{K}^2\,(]0,+\infty[)$. Our aim is now to show that the operators $V(t)$ act on E_m^0 as well and $V(t) = T_m(t)$ on E_m^0 for every $t > 0$, where $(T_m(t))_{t \geqslant 0}$ is the C_0-semigroup considered in Theorem 3.1 for $p(x) = x^2$ $(x \geqslant 0)$.

We shall proceed in several steps. First of all we point out that, if $f \in E_m^0$, then for every $t > 0$ and $x \geqslant 0$ the integral

$$V(t)f(x) = \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{+\infty} f(xe^{u-t/2}) e^{-\frac{u^2}{2t}} \, du$$

is absolutely convergent because

$$|f(xe^{u-t/2})e^{-\frac{u^2}{2t}}| \leqslant \|f\|_m(1+x^m e^{m(u-t/2)})e^{-u^2/2t} \quad (u \in \mathbb{R}).$$

Proposition 4.1. *Let $m \geqslant 1$. For every $t > 0$, $V(t)$ is a bounded linear operator from E_m^0 into E_m^0 and $\|V(t)\| = e^{m(m-1)t/2}$. Moreover, $\lim_{t\to 0^+} V(t)f = f$ in E_m^0 for every $f \in E_m^0$.*

Proof. Let $f \in E_m^0$. We have first to show that $V(t)f \in E_m^0$. It is easy to show that $V(t)f$ is continuous by using the Lebesgue's dominated convergence theorem, the continuity of f and the uniform estimate

$$|f(xe^{u-t/2})e^{-u^2/2t}| \leqslant \|f\|_m(1+b^m e^{m(u-t/2)})e^{-u^2/2t} \tag{1}$$

which holds true for every $u \in \mathbb{R}$ and $x \in [0, b]$, and for every $b > 0$.

In order to evaluate the asymptotic behaviour of

$$\frac{V(t)f(x)}{1+x^m} = \frac{1}{\sqrt{2\pi t}}\int_{-\infty}^{+\infty} \frac{f(xe^{u-t/2})}{1+x^m} e^{-u^2/2t}\, du \tag{2}$$

note that, for every $u \in \mathbb{R}$,

$$\frac{|f(xe^{u-t/2})|}{1+x^m} \leqslant \|f\|_m \frac{1+x^m e^{m(u-t/2)}}{1+x^m} \leqslant \|f\|_m max\{1, e^{m(u-t/2)}\}, \tag{3}$$

so that the absolute value of the integrand in (2) is majorized by $\|f\|_m \varphi$ where

$$\varphi(u) := sup\left\{\frac{e^{-u^2/2t}}{\sqrt{2\pi t}}, \frac{e^{m(u-t/2)-u^2/2t}}{\sqrt{2\pi t}}\right\} \quad (u \in \mathbb{R}) \tag{4}$$

and φ is Lebesgue integrable on $\mathbb{R}$. So, by the Lebesgue's dominated convergence theorem

$$\lim_{x\to+\infty} \frac{V(t)f(x)}{1+x^m} = 0$$

and hence $V(t)f \in E_m^0$. To show that $V(t)$ is bounded, we first point out that

$$\begin{aligned}
&\frac{1}{\sqrt{2\pi t}}\int_{-\infty}^{+\infty} \frac{1+x^m e^{m(u-t/2)}}{1+x^m} e^{-u^2/2t}\, du \\
&= \frac{1}{1+x^m}\left[\frac{1}{\sqrt{2\pi t}}\int_{-\infty}^{+\infty} e^{-u^2/2t}\, du + x^m e^{m(m-1)t/2}\frac{1}{\sqrt{2\pi t}}\int_{-\infty}^{+\infty} e^{-\frac{(u-mt)^2}{2t}}\, du\right] \\
&= \frac{1+x^m e^{m(m-1)t/2}}{1+x^m} \leqslant e^{m(m-1)t/2}.
\end{aligned}$$

Hence from the first inequality in (3) it follows that

$$\|V(t)f\|_m \leqslant e^{m(m-1)t/2}\|f\|_m \quad \text{so that} \quad \|V(t)\| \leqslant e^{m(m-1)t/2}.$$

On the other hand, for every real number $\lambda \in [0, m[$, considering the function $e_\lambda(x) := x^\lambda \quad (x \geqslant 0)$, we have

$$V(t)e_\lambda(x) = \frac{x^\lambda e^{-\lambda t/2}}{\sqrt{2\pi t}}\int_{-\infty}^{+\infty} e^{\lambda u} e^{-u^2/2t}\, du = x^\lambda e^{\lambda(\lambda-1)t/2}. \tag{5}$$

Therefore $V(t)e_\lambda = e^{\lambda(\lambda-1)t/2}e_\lambda$ and hence $\|V(t)\| \geqslant e^{\lambda(\lambda-1)t/2}$.
Letting $\lambda \to m$, we get $\|V(t)\| \geqslant e^{m(m-1)t/2}$ and so we obtain the desired equality. As regards the last part of the statement, chosen $\lambda \in]0, 1/2[$, from (5) it follows that

$$\lim_{t\to 0^+} V(t)e_\lambda = e_\lambda \quad \text{and} \quad \lim_{t\to 0^+} V(t)e_{2\lambda} = e_{2\lambda}$$

in E_m^0 and, of course, $\lim_{t\to 0^+} V(t)\mathbf{1} = \mathbf{1}$. Since $(V(t))_{0<t\leqslant 1}$ is equibounded and $\{\mathbf{1}, e_\lambda, e_{2\lambda}\}$ is a Korokvin set in E_m^0(see [5, Lemma 4.1]), we have that $\lim_{t\to 0^+} V(t)f = f$ in E_m^0 for every $f \in E_m^0$. $\square$

A further property of the operators $V(t)$ is indicated below.
Recall that

$$\mathcal{K}^2\,(]0, +\infty[) := \{f \in \mathcal{C}^2\,(]0, +\infty[) \mid f \text{ has compact support}\}.$$

Clearly, $\mathcal{K}^2\,(]0, +\infty[) \subset D_m(A)$ $(m \geqslant 1)$ where $D_m(A)$ is defined by (3.2), with $p(x) = x^2$. Furthermore, every $f \in \mathcal{K}^2\,(]0, +\infty[)$ can be obviously extended to a function in $\mathcal{K}^2(\mathbb{R})$.

Proposition 4.2. *Let $m \geqslant 1$. Then for every $t > 0$*

$$V(t)(\mathcal{K}^2\,(]0, +\infty[)) \subset D_m(A).$$

Proof. Fix $t > 0$ and $f \in \mathcal{K}^2\,(]0, +\infty[)$. For simplicity write

$$V(t)f(x) = \int_{-\infty}^{+\infty} \varphi(x, u)\,du \quad (x \geqslant 0) \tag{1}$$

where

$$\varphi(x, u) := \frac{1}{\sqrt{2\pi t}} f(xe^{u-t/2})e^{-u^2/2t} \quad (x \geqslant 0,\, u \in \mathbb{R}). \tag{2}$$

Then

$$\begin{aligned}\left|\frac{\partial}{\partial x}\varphi(x, u)\right| &= \frac{1}{\sqrt{2\pi t}}|f'(xe^{u-t/2})|e^{u-t/2}e^{-u^2/2t} \\ &\leqslant \frac{\|f'\|_\infty}{\sqrt{2\pi t}}e^{-u^2/2t+u-t/2} =: g_1(u)\end{aligned}$$

and $g_1 \in \mathcal{L}^1(\mathbb{R})$. Analogously

$$\begin{aligned}\left|\frac{\partial^2}{\partial x^2}\varphi(x, u)\right| &= \frac{1}{\sqrt{2\pi t}}|f''(xe^{u-t/2})|e^{2u-t}e^{-u^2/2t} \\ &\leqslant \frac{\|f''\|_\infty}{\sqrt{2\pi t}}e^{-u^2/2t+2u-t} =: g_2(u)\end{aligned}$$

with $g_2 \in \mathcal{L}^1(\mathbb{R})$. So it is possible to differentiate under the sign of the integral and

$$D^2(V(t)f)(x) = \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{+\infty} f''(xe^{u-t/2})e^{2u-t}e^{-u^2/2t}\,du \quad (x \geqslant 0).$$

Since f'' is continuous and bounded, from the Lebesgue's dominated convergence theorem it follows that $D^2(V(t)f)$ is continuous on $[0, +\infty[$.

It remains to show that the two boundary conditions defining $D_m(A)$ are satisfied. The first one is obvious. As regards the second one, for every $x > 0$ we have

$$\frac{x^2 D^2(V(t)f)(x)}{1+x^m} = \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{+\infty} \frac{x^2 f''(xe^{u-t/2})}{1+x^m} e^{2u-t} e^{-u^2/2t}\, du. \tag{3}$$

Now, the integrand in (3) goes to 0 as $x \to +\infty$. Furthermore, there exists $M \geqslant 0$ such that $x^2|f''(x)| \leqslant M$ $(x \geqslant 0)$ and hence, for $x \geqslant 0$ and $u \in \mathbb{R}$,

$$\frac{x^2|f''(xe^{u-t/2})|}{1+x^m} e^{2u-t} e^{-u^2/2t} \leqslant M e^{-u^2/2t}.$$

Again from the Lebesgue's dominated convergence theorem it follows that $\lim_{x\to+\infty} \frac{x^2 D^2(V(t)f)(x)}{1+x^m} = 0$ and the proof is now complete. □

Proposition 4.3. *For every $m \geqslant 1$ and $t > 0$, $V(t) = T_m(t)$ on $\mathcal{K}^2\ (]0, +\infty[)$.*

Proof. Fix $m \geqslant 1$ and $f \in \mathcal{K}^2\ (]0, +\infty[) \subset D_m(A)$. Set

$$u(x,t) := V(t)f(x) \quad (x \geqslant 0, t > 0).$$

Then $u(\cdot, t) \in D_m(A)$ by Proposition 4.2 and u solves the Cauchy problem

$$\begin{cases} \frac{\partial u}{\partial t}(x,t) = \frac{x^2}{2}\frac{\partial^2 u}{\partial x^2}(x,t) & (x \geqslant 0, t > 0), \\ \lim\limits_{t\to 0^+} u(\cdot, t) = f & \text{in } E_m^0, \end{cases}$$

by virtue of Proposition 4.1.
Therefore $u(x,t) = T_m(t)f(x) \quad (x \geqslant 0, t > 0)$ and hence the result follows. □

We are now in the position to show our main result.

Theorem 4.4. *Let $(T_m(t))_{t\geqslant 0}$ be the C_0-semigroup generated by $(A, D_m(A))$ in E_m^0 $(m \geqslant 1)$. Then for every $t > 0$, $f \in E_m^0$ and $x \geqslant 0$,*

$$T_m(t)f(x) = V(t)f(x) = \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{+\infty} f(xe^{u-t/2}) e^{-u^2/2t}\, du.$$

Furthermore,

$$\|T_m(t)\| = e^{m(m-1)t/2}.$$

Proof. Since $V(t)\mathbf{1} = \mathbf{1} = T_m(t)\mathbf{1}$, it is enough to show that $T_m(t)$ and $V(t)$ coincide on $\tilde{E}_m^0 := \{f \in E_m^0 \mid f(0) = 0\}$. This, in turn, will follow from Proposition 4.3 if we prove that $\mathcal{K}^2\ (]0, +\infty[)$ is dense in $(\tilde{E}_m^0, \|\cdot\|_m)$, because the operators $V(t)$ and $T_m(t)$ are bounded on E_m^0.

To this end, note that $\widetilde{E_m^0}$ is isometrically isomorphic to the space $(\mathcal{C}_0\,(]0,+\infty[)\,,\|\cdot\|_\infty)$ by means of the isomorphism $\sigma : \widetilde{E_m^0} \longrightarrow \mathcal{C}_0\,(]0,+\infty[)$ defined by

$$\sigma(f) := w_m f \quad (f \in \widetilde{E_m^0}).$$

So it is enough to remark that $\sigma(\mathcal{K}^2\,(]0,+\infty[)) = \mathcal{K}^2\,(]0,+\infty[)$ is dense in $(\mathcal{C}_0\,(]0,+\infty[)\,,\|\cdot\|_\infty)$.

The last equality follows from Proposition 4.1. □

We end the paper by investigating the asymptotic behaviour of the semigroups $(T_m(t))_{t\geqslant 0}$ on $\mathcal{C}_b\,([0,+\infty[)$. However, note that, by Remark 3.2, $T_m(t) = T_1(t)$ on $\mathcal{C}_b\,([0,+\infty[)$ for every $m\geqslant 1$ and $t\geqslant 0$.
From the general theory it is known that the solution X_t^x of (4.2) satisfies for all $x > 0$

$$\lim_{t\to+\infty} X_t^x = 0, \quad \text{a.s.}$$

(See [9, Exercise 5.31, p. 349].)

This means that for $f \in \mathcal{C}_b\,([0,+\infty[)$ one has

$$\lim_{t\to+\infty} f(X_t^x) = f(0), \quad \text{a.s.}$$

and, by the dominated convergence theorem,

$$\lim_{t\to+\infty} Ef(X_t^x) = Ef(0) = f(0).$$

This yields

$$\lim_{t\to+\infty} T_1(t)f(x) = f(0).$$

We shall give an analytical proof of this fact. Note, however, that this result cannot be valid in the other spaces E_m^0, $m\geqslant 1$, because of formula (5) in the proof of Proposition 4.1.

Theorem 4.5. *For every $f \in \mathcal{C}_b\,([0,+\infty[)$ and $x\geqslant 0$,*

$$\lim_{t\to+\infty} T_1(t)f(x) = f(0).$$

Proof. If $x = 0$, the result is obvious. Assume $x > 0$. We have $|f(s)|\leqslant M,\ s \in [0,+\infty)$, for some constant $M > 0$.
Let $\varepsilon > 0$. There exists $\delta > 0$ such that

$$|f(s) - f(0)|\leqslant \frac{\varepsilon}{2}, \quad s \in [0,\delta].$$

Moreover, there exists $A > 0$ such that

$$t^{3/4} - \frac{t}{2} \leqslant \log\frac{\delta}{x}, \quad t\geqslant A.$$

Let $t \geqslant max\{A, 16M^2/\varepsilon^2\}$. Then

$$|T_1(t)f(x) - f(0)| \leqslant \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{+\infty} |f(xe^{u-t/2}) - f(0)| e^{-u^2/2t}\, du$$
$$= \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{\log \frac{\delta}{x}} |f(xe^v) - f(0)| e^{-(v+t/2)^2/2t}\, dv + \frac{1}{\sqrt{2\pi t}} \int_{\log \frac{\delta}{x}}^{\infty} |f(xe^v) - f(0)| e^{-(v+t/2)^2/2t}\, dv.$$

For $v \leqslant \log \frac{\delta}{x}$ we have $xe^v \leqslant \delta$, so that

$$\frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{\log \frac{\delta}{x}} |f(xe^v) - f(0)| e^{-(v+t/2)^2/2t}\, dv \leqslant \frac{\varepsilon}{2} \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{+\infty} e^{-(v+t/2)^2/2t}\, dv = \frac{\varepsilon}{2}.$$

On the other hand,

$$\frac{1}{\sqrt{2\pi t}} \int_{\log \frac{\delta}{x}}^{\infty} e^{-(v+t/2)^2/2t}\, dv \leqslant \frac{1}{\sqrt{2\pi t}} \int_{t^{3/4}-t/2}^{\infty} e^{-(v+t/2)^2/2t}\, dv$$
$$= \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{\infty} e^{-(v+t/2)^2/2t} \mathbf{1}_{\{v \geqslant t^{3/4}-t/2\}}\, dv$$
$$\leqslant \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{\infty} e^{-(v+t/2)^2/2t} t^{-3/2} \left(v + \frac{t}{2}\right)^2 dv$$
$$= t^{-3/2} \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{\infty} u^2 e^{-u^2/2t}\, du = t^{-3/2} t = t^{-1/2}.$$

Thus

$$\frac{1}{\sqrt{2\pi t}} \int_{\log \frac{\delta}{x}}^{\infty} |f(xe^v) - f(0)| e^{-(v+t/2)^2/2t}\, dv \leqslant 2Mt^{-1/2} \leqslant \frac{\varepsilon}{2}.$$

In conclusion, $|T_1(t)f(x) - f(0)| \leqslant \varepsilon$, and the proof is complete. □

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